

LOG-SOBOLEV, ISOPERIMETRY AND TRANSPORT INEQUALITIES ON GRAPHS

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ABSTRACT. In this paper, we study some functional inequalities (such as Poincaré inequalities, logarithmic Sobolev inequalities, generalized Cheeger isoperimetric inequalities, transportation-information inequalities and transportation-entropy inequalities) for reversible nearest-neighbor Markov processes on a connected finite graph by means of (random) path method. We provide estimates of the involved constants.

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1. INTRODUCTION

Let $G = (V, E)$ be a finite connected graph with vertex set V and oriented edges set E , which is a symmetric subset of $V^2 \setminus \{(x, x); x \in V\}$. If $(x, y) \in E$, we call that x, y are adjacent, denoted by $x \sim y$. Consider the operator

$$\mathcal{L}f(x) = \sum_y q(x, y) (f(y) - f(x)), \quad \text{for all } x \in V \quad (1.1)$$

for any function $f : V \rightarrow \mathbb{R}$, where the jump rate $q(x, y)$ from x to y is non-negative, and $q(x, y) > 0$ if and only if $x \sim y$.

Let (X_t) be the Markov process generated by \mathcal{L} , defined on $(\Omega, (\mathcal{F}_t)_{t \geq 0}, (\mathbb{P}_x)_{x \in V})$. We always assume the *reversibility* condition, i.e. there is some probability measure μ satisfying the *detailed balance condition*

$$Q(x, y) := \mu(x)q(x, y) = \mu(y)q(y, x), \quad \forall (x, y) \in E. \quad (1.2)$$

Equivalently, the operator \mathcal{L} is self-adjoint on $L^2(\mu)$, that is

$$\begin{aligned} \langle f, -\mathcal{L}g \rangle_\mu &= \langle -\mathcal{L}f, g \rangle_\mu = \frac{1}{2} \sum_{x, y} (f(x) - f(y)) (g(x) - g(y)) Q(x, y) \\ &= \frac{1}{2} \sum_{\mathbf{e} \in E} D_{\mathbf{e}} f D_{\mathbf{e}} g Q(\mathbf{e}) =: \mathcal{E}(f, g), \end{aligned}$$

where $D_{\mathbf{e}} f := f(y) - f(x)$ for $\mathbf{e} = (x, y) \in E$. When $q(x, y) = 1/d_x$ with d_x the *degree* of x (the number of neighbors $y \sim x$), \mathcal{L} becomes the *Laplacian* Δ on the graph. In that case $\mu(x) = d_x/|E|$ and $Q(\mathbf{e}) = 1/|E|$.

Define the variance of f with respect to μ

$$\text{Var}_\mu(f) = \mu((f - \mu(f))^2),$$

and the entropy of f^2 under μ

$$\text{Ent}_\mu(f^2) = \mu(f^2 \log f^2) - \mu(f^2) \log \mu(f^2).$$

We say that μ satisfies a Poincaré inequality if there exists a constant $\lambda > 0$ such that for all $f \in L^2(\mu)$,

$$\text{Var}_\mu(f) \leq \lambda \mathcal{E}(f, f), \quad (1.3)$$

μ satisfies a log-Sobolev inequality if there exists a constant $\alpha > 0$ such that for all $f \in L^2(\mu)$,

$$\text{Ent}_\mu(f^2) \leq 2\alpha \mathcal{E}(f, f). \quad (1.4)$$

The optimal constants λ and α in (1.3) and (1.4) are called respectively the Poincaré constant and the log-Sobolev constant of μ , which are denoted by c_P and c_{LS} respectively. It is well known that $c_P \leq c_{LS}$, see [25].

The Poincaré inequality and logarithmic Sobolev inequality play a crucial role in the analysis of the behaviour of the process. To study of the Poincaré constant, the path combinatoric method was introduced by Jerrum and Sinclair [19] in theoretic computer science, and further developed by Diaconis and Stroock [13], Fill [15], Sinclair [29], Chen [7], and so on. The logarithmic Sobolev inequality in the discrete setting was also studied by many authors, such as Diaconis and Saloff-Coste [12], Roberto [24], Lee and Yau [22], Chen [8], Chen and Sheu [6], Chen *et al.* [5], and so on. The reader is referred to the books of Saloff-Coste [27] and Chen [9] for further information.

The main purpose of this paper is to study the logarithmic Sobolev inequality, the generalized Cheeger isoperimetric inequality and the transport inequality.

The remainder of this paper is organized as follows: in next section, we focus on the logarithmic Sobolev inequality, the third section is devoted to the transportation-information inequality and the generalized Cheeger isoperimetric inequality. In the last section, some examples are discussed and the estimates of involved constants are given.

2. LOGARITHMIC SOBOLEV INEQUALITY

2.1. Length functions, random paths. A path γ_{xy} from x to y is a family of edges $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ where $\mathbf{e}_k = (x_{k-1}, x_k) \in E$, such that $x_0 = x, x_n = y$. It is said to have no circle if all $x_k, k = 0, \dots, n$, are different. A positive function $w : E \rightarrow (0, +\infty)$ defined on the edge set E is called *length function*, if $w(x, y) = w(y, x)$ for any $\mathbf{e} = (x, y) \in E$. Given the length function w , the *w-length* of a path γ_{xy} from x to y is defined by

$$|\gamma_{xy}|_w := \sum_{\mathbf{e} \in \gamma_{xy}} w(\mathbf{e}),$$

and the distance associated with w is

$$\rho_w(x, y) := \min_{\gamma_{xy}} |\gamma_{xy}|_w.$$

When $w \equiv 1$, $\rho_w =: \rho_1$ is the natural graph distance on V .

Diaconis and Stroock [13] showed that the Poincaré constant c_P satisfies that

$$c_P \leq \max_{\mathbf{e} \in E} \sum_{x, y \in V} 1_{\gamma_{xy}}(\mathbf{e}) |\gamma_{xy}|_{1/Q} \mu(x) \mu(y), \quad (2.1)$$

for any collection of paths $\{\gamma_{xy}, x, y \in V\}$, where $|\gamma_{xy}|_{1/Q}$ is the length $|\gamma_{xy}|_w$ with $w(\mathbf{e}) = 1/Q(\mathbf{e})$, a quite natural distance associated with the Markov process. Furthermore, by using the length functions, the estimate (2.1) can be improved to be

$$c_P \leq \max_{\mathbf{e} \in E} \frac{1}{Q(\mathbf{e})w(\mathbf{e})} \sum_{x, y \in V} 1_{\gamma_{xy}}(\mathbf{e}) |\gamma_{xy}|_w \mu(x) \mu(y), \quad (2.2)$$

which is sharp for birth-death processes (see Kahale [20] or Chen [7]).

Now for any $x, y \in V$ different, let γ_{xy} be a random (maybe deterministic) path without circle from x to y . By convention we set $\gamma_{xx} = \emptyset$, and denote by \mathbb{E}^γ the expectation w.r.t. $\{\gamma_{xy}, x, y \in V\}$.

2.2. Logarithmic Sobolev inequality.

Theorem 2.1. *For any length function w and any edge \mathbf{e} , let*

$$L_{w, \mathbf{e}}(x) := \mathbb{E}^\gamma \sum_{y \in V} 1_{\gamma_{xy}}(\mathbf{e}) |\gamma_{xy}|_w \mu(y). \quad (2.3)$$

The logarithmic Sobolev constant c_{LS} is bounded by

$$c_{LS} \leq \inf_w \max_{\mathbf{e} \in E} \frac{1}{Q(\mathbf{e})w(\mathbf{e})} \left(\text{Ent}_\mu(L_{w, \mathbf{e}}) + \mu(L_{w, \mathbf{e}}) \log(e^2 + 1) \right), \quad (2.4)$$

where \inf_w is taken over all length functions w on E and e is the Euler constant.

The upper bound in (2.4) gives us a very practical criterion for the logarithmic Sobolev inequality, following the classical idea of Lyapunov function method for stability.

The logarithmic Sobolev inequality above is based on the following weighted Poincaré inequality, which is a slight generalization of (2.2).

Lemma 2.2 (Weighted Poincaré inequality). *Let φ be a nonnegative function on V , then for any length function w ,*

$$\sum_{x \in V} (f(x) - \mu(f))^2 \varphi(x) \mu(x) \leq c(\varphi, w) \mathcal{E}(f, f), \quad \forall f : V \rightarrow \mathbb{R}, \quad (2.5)$$

where

$$\begin{aligned} c(\varphi, w) &:= \max_{\mathbf{e} \in E} \frac{2}{Q(\mathbf{e})w(\mathbf{e})} \mathbb{E}^\gamma \sum_{x, y \in V} 1_{\gamma_{xy}}(\mathbf{e}) |\gamma_{xy}|_w \varphi(x) \mu(x) \mu(y) \\ &= \max_{\mathbf{e} \in E} \frac{2}{Q(\mathbf{e})w(\mathbf{e})} \sum_{x \in V} L_{w, \mathbf{e}}(x) \varphi(x) \mu(x). \end{aligned}$$

When $\varphi \equiv 1$, our constant $c(\varphi, w)$ is twice of the quantity at the right hand side of (2.2).

Proof. For any fixed realization of random paths $\{\gamma_{xy}; x, y \in V\}$,

$$\begin{aligned}
& \sum_{x \in V} (f(x) - \mu(f))^2 \varphi(x) \mu(x) \\
&= \sum_{x \in V} \left(\sum_{y \in V} (f(x) - f(y)) \mu(y) \right)^2 \varphi(x) \mu(x) \\
&= \sum_{x \in V} \left(\sum_{y \in V} \mu(y) \sum_{\mathbf{e} \in \gamma_{xy}} D_{\mathbf{e}} f \right)^2 \varphi(x) \mu(x) \\
&\leq \sum_{x, y \in V} \left(\sum_{\mathbf{e} \in \gamma_{xy}} D_{\mathbf{e}} f \right)^2 \varphi(x) \mu(x) \mu(y) \\
&\leq \sum_{x, y \in V} \left(\sum_{\mathbf{e} \in \gamma_{xy}} w(\mathbf{e}) \right) \left(\sum_{\mathbf{e} \in \gamma_{xy}} \frac{1}{w(\mathbf{e})} (D_{\mathbf{e}} f)^2 \right) \varphi(x) \mu(x) \mu(y) \\
&= \frac{1}{2} \sum_{\mathbf{e} \in E} Q(\mathbf{e}) (D_{\mathbf{e}} f)^2 \cdot \frac{2}{Q(\mathbf{e}) w(\mathbf{e})} \sum_{x, y \in V} 1_{\gamma_{xy}}(\mathbf{e}) |\gamma_{xy}|_w \varphi(x) \mu(x) \mu(y),
\end{aligned}$$

where the Cauchy-Schwarz inequality is applied twice. Taking the expectation \mathbb{E}^γ w.r.t. the randomness of γ , we get the desired result. \square

Now recall two important lemmas: the first one is due to Rothaus [26] and the second was given by Barthe-Roberto [2].

Lemma 2.3. For any real function f on V and any constant $a \in \mathbb{R}$,

$$\text{Ent}_\mu(f^2) \leq \text{Ent}_\mu((f - a)^2) + 2\mu((f - a)^2).$$

Lemma 2.4. For any real function f on V ,

$$\text{Ent}_\mu(f^2) + 2\mu(f^2) \leq \sup \left\{ \mu(f^2 \varphi); \varphi \geq 0, \mu(e^\varphi) \leq e^2 + 1 \right\}.$$

Consequently, by Donsker-Varadhan's variational formula (see [14]),

$$\mu(f^2 \varphi) - \mu(f^2) \log \mu(e^\varphi) \leq \text{Ent}_\mu(f^2), \quad \forall \varphi,$$

we have

$$\sup \left\{ \mu(f^2 \varphi); \varphi \geq 0, \mu(e^\varphi) \leq e^2 + 1 \right\} \leq \text{Ent}_\mu(f^2) + \mu(f^2) \log(e^2 + 1). \quad (2.6)$$

Proof of Theorem 2.1. By Lemma 2.3,

$$\begin{aligned}
\text{Ent}_\mu(f^2) &\leq \text{Ent}_\mu((f - \mu(f))^2) + 2\mu((f - \mu(f))^2) \\
&\leq \sup \left\{ \sum_{x \in V} (f(x) - \mu(f))^2 \varphi(x) \mu(x); \varphi \geq 0, \mu(e^\varphi) \leq e^2 + 1 \right\} \\
&\leq \sup_{\varphi \geq 0, \mu(e^\varphi) \leq e^2 + 1} c(\varphi, w) \cdot \mathcal{E}(f, f),
\end{aligned}$$

where the last inequality follows from (2.5). Moreover by (2.5) and (2.6), we have

$$\begin{aligned} \sup_{\varphi \geq 0, \mu(e^\varphi) \leq e^2+1} c(\varphi, w) &= \max_{\mathbf{e} \in E} \frac{2}{Q(\mathbf{e})w(\mathbf{e})} \sup_{\varphi \geq 0, \mu(e^\varphi) \leq e^2+1} \mu(L_{w,\mathbf{e}}\varphi) \\ &\leq \max_{\mathbf{e} \in E} \frac{2}{Q(\mathbf{e})w(\mathbf{e})} (\text{Ent}_\mu(L_{w,\mathbf{e}}) + \mu(L_{w,\mathbf{e}}) \log(e^2 + 1)), \end{aligned}$$

where (2.4) follows. \square

3. TRANSPORTATION INEQUALITIES

In this section, we shall establish the transportation-information inequality W_1I and as a corollary, the transportation-entropy inequality W_1H . For this purpose, let us introduce some notions and notations.

3.1. Wasserstein distance, entropy and information. Given a metric ρ on V , the Lipschitzian norm of a function g is denoted by $\|g\|_{\text{Lip}(\rho)}$. For two probability measures ν, μ on V , say $\nu, \mu \in \mathcal{M}_1(V)$,

(i) their Wasserstein distance $W_{1,\rho}(\nu, \mu)$ associated with ρ is defined as

$$W_{1,\rho}(\nu, \mu) = \inf_{\pi} \iint_{V^2} \rho(x, y) \pi(dx, dy),$$

where π runs over all couplings of (ν, μ) , i.e., probability measures on V^2 such that $\pi(A \times V) = \nu(A)$ and $\pi(V \times A) = \mu(A)$ for all Borel subsets A of V . If $\rho(x, y) = 1_{x \neq y}$ is the discrete metric, $W_{1,\rho}(\nu, \mu) = \frac{1}{2} \|\nu - \mu\|_{\text{TV}}$ where $\|\tilde{\nu}\|_{\text{TV}} = \sup_{|f| \leq 1} \tilde{\nu}(f)$ is the total variation of a signed measure $\tilde{\nu}$.

(ii) The relative entropy of ν w.r.t. μ is given by

$$H(\nu|\mu) = \begin{cases} \sum_{x \in V} \nu(x) \log \frac{\nu(x)}{\mu(x)}, & \text{if } \nu \ll \mu; \\ +\infty, & \text{otherwise.} \end{cases}$$

(iii) Fisher-Donsker-Varadhan information of a probability $\nu = h^2\mu$ w.r.t. μ is defined by

$$I(\nu|\mu) := \frac{1}{2} \sum_{x, y \in V} (h(x) - h(y))^2 Q(x, y) = \frac{1}{2} \sum_{\mathbf{e} \in E} (D_{\mathbf{e}}h)^2 Q(\mathbf{e}),$$

where $D_{\mathbf{e}}h = h(y) - h(x)$ for the oriented edge $\mathbf{e} = (x, y) \in E$.

3.2. Transportation-information inequality. Guillin *et al.* [17] introduced the following transportation-information inequality for the given metric ρ ,

$$W_{1,\rho}^2(\nu, \mu) \leq 2c_G I(\nu|\mu), \quad \forall \nu \in \mathcal{M}_1(V), \quad (3.1)$$

where c_G is the best constant. In [17], it is proved that (3.1) is equivalent to the following Gaussian concentration inequality: for all probabilities $\nu \ll \mu$ and ρ -Lipschitzian function g on V ,

$$\mathbb{P}_\nu \left(\frac{1}{t} \int_0^t g(X_s) ds > \mu(g) + r \right) \leq \left\| \frac{d\nu}{d\mu} \right\|_{L^2} \exp \left\{ -\frac{tr^2}{2c_G \|g\|_{\text{Lip}(\rho)}^2} \right\}, \quad \forall t, r > 0. \quad (3.2)$$

Here (X_t) is the Markov process generated by \mathcal{L} , defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with initial distribution ν . So we call c_G the Gaussian concentration constant for (X_t) (associated to the metric ρ).

The reader is referred to the book of Villani [30] for optimal transport, transport inequalities and related bibliographies.

Theorem 3.1. *The transportation-information inequality (3.1) holds with*

$$c_G \leq K := \inf_w K(w), \quad (3.3)$$

where the infimum is taken over all length functions w and the geometric constant $K(w)$ is given by

$$K(w) = \max_{\mathbf{e} \in E} \frac{1}{Q(\mathbf{e})w(\mathbf{e})} \sum_{x,y \in V} \mathbb{E}^\gamma 1_{\gamma_{xy}}(\mathbf{e}) \rho^2(x,y) |\gamma_{xy}|_w \mu(x) \mu(y). \quad (3.4)$$

Remark 3.2. When $\rho(x,y) = 1_{x \neq y}$ (the discrete metric), K coincides with the quantity in (2.2). By Guillin *et al.* [17, Theorem 3.1], the transportation-information inequality w.r.t. the discrete metric and the Poincaré inequality are equivalent:

$$\frac{c_P}{8} \leq c_G \leq 2c_P.$$

If we apply this result together with (2.2), we obtain only $c_G \leq 2K$. Since $c_P = K$ for birth-death processes (see [20, 7]), we get $c_G \geq K/8$. In other words, our estimate of c_G is of correct order.

Proof of Theorem 3.1. For each probability measure $\nu = h^2 \mu$, by Kantorovich-Robinstein's identity (see [30]) and the Cauchy-Schwarz inequality,

$$\begin{aligned} W_{1,\rho}(\nu, \mu) &= \sup_{\|g\|_{\text{Lip}(\rho)} \leq 1} \sum_{x \in V} g(x) (h^2(x) - 1) \mu(x) \\ &= \frac{1}{2} \sup_{\|g\|_{\text{Lip}(\rho)} \leq 1} \sum_{x,y \in V} (g(x) - g(y)) (h^2(x) - h^2(y)) \mu(x) \mu(y) \\ &\leq \frac{1}{2} \sup_{\|g\|_{\text{Lip}(\rho)} \leq 1} \sqrt{\sum_{x,y \in V} (g(x) - g(y))^2 (h(x) - h(y))^2 \mu(x) \mu(y)} \\ &\quad \cdot \sqrt{\sum_{x,y} (h(x) + h(y))^2 \mu(x) \mu(y)} \\ &\leq \sqrt{\sum_{x,y \in V} \rho^2(x,y) (h(x) - h(y))^2 \mu(x) \mu(y)}. \end{aligned}$$

For any fixed random paths $\{\gamma_{xy} = \gamma_{xy}(\omega); x, y \in V\}$ and the length function w , we have by the Cauchy-Schwarz inequality,

$$\begin{aligned} \sum_{x,y} \rho^2(x,y) (h(x) - h(y))^2 \mu(x)\mu(y) &= \sum_{x,y} \rho^2(x,y) \left(\sum_{\mathbf{e} \in \gamma_{x,y}} D_{\mathbf{e}} h \right)^2 \mu(x)\mu(y) \\ &\leq \sum_{x,y} \rho^2(x,y) \mu(x)\mu(y) \left(\sum_{\mathbf{e} \in \gamma_{xy}} (D_{\mathbf{e}} h)^2 \frac{1}{w(\mathbf{e})} \right) \left(\sum_{\mathbf{e} \in \gamma_{xy}} w(\mathbf{e}) \right) \\ &= \sum_{\mathbf{e} \in E} (D_{\mathbf{e}} h)^2 Q(\mathbf{e}) \cdot \frac{1}{Q(\mathbf{e})w(\mathbf{e})} \sum_{x,y \in V} 1_{\gamma_{x,y}}(\mathbf{e}) \rho^2(x,y) |\gamma_{xy}|_w \mu(x)\mu(y). \end{aligned}$$

Taking first the expectation \mathbb{E}^γ and then the maximum of the last term over all oriented edges \mathbf{e} , we get $c_G \leq K(w)$, the desired result. \square

Corollary 3.3. Assume that there exists some constant $M > 0$ such that

$$\sup_{x \in V} \frac{1}{2} \sum_{y \sim x} \rho^2(x,y) q(x,y) \leq M. \quad (3.5)$$

Then the following transportation-entropy inequality holds

$$W_{1,\rho}^2(\nu, \mu) \leq \sqrt{2KM} H(\nu|\mu), \quad \forall \nu \in \mathcal{M}_1(V), \quad (3.6)$$

or equivalently for any Lipschitzian function g ,

$$\int e^{\lambda(g - \mu(g))} d\mu \leq \exp \left(\frac{\lambda^2 \sqrt{2MK}}{4} \|g\|_{\text{Lip}(\rho)}^2 \right), \quad \lambda \in \mathbb{R}. \quad (3.7)$$

Proof. The transportation-entropy inequality (3.6) follows from the transportation information inequality (3.3) under the condition (3.5), by Guillin *et al.* [18, Theorem 4.2]. The equivalence between (3.6) and the Gaussian concentration (3.7) is the famous Bobkov-Götze's characterization in [3]. \square

Corollary 3.4. For the Laplacian $\mathcal{L} = \Delta$ on the connected graph $G = (V, E)$, we have for the graph metric ρ_1 ,

$$c_G \leq K \leq \frac{d_*^2 b D^3}{|E|},$$

where $d_* = \max_{x \in V} d_x$, D is the diameter of G and

$$b = \max_{\mathbf{e}} \sharp\{\rho_1\text{-shortest paths } \gamma : \mathbf{e} \in \gamma\}. \quad (3.8)$$

Proof. Choose γ_{xy} distributed uniformly on all shortest paths from x to y and $w = 1$, we see that K is bounded from above by (noting that $Q(\mathbf{e}) = 1/|E|$, $\mu(x) = d_x/|E|$)

$$\max_{\mathbf{e} \in E} |E| \mathbb{E}^\gamma \sum_{x,y \in V} 1_{\gamma_{xy}}(\mathbf{e}) \rho_1(x,y)^3 \left(\frac{d_*}{|E|} \right)^2 \leq \frac{d_*^2 b D^3}{|E|}.$$

\square

3.3. Generalized Cheeger isoperimetric inequality. Consider the following generalized Cheeger isoperimetric inequality

$$W_{1,\rho}(f\mu, \mu) \leq \frac{c_1}{2} \sum_{\mathbf{e} \in E} |D_{\mathbf{e}}f| Q(\mathbf{e}), \quad \forall \nu = f\mu \in \mathcal{M}_1(V), \quad (3.9)$$

where c_1 is the best constant, called as *Cheeger constant* w.r.t. the metric ρ .

Define the geometric constant κ

$$\kappa := \max_{\mathbf{e} \in E} \frac{1}{Q(\mathbf{e})} \mathbb{E}^\gamma \sum_{x,y \in V} 1_{\gamma_{xy}}(\mathbf{e}) \rho(x, y) \mu(x) \mu(y). \quad (3.10)$$

Theorem 3.5. *It holds that*

$$c_1 \leq \kappa.$$

Proof. By Kantorovich-Robinstein's identity, we have for any fixed random paths $\{\gamma_{xy}; x, y \in V\}$,

$$\begin{aligned} W_{1,\rho}(f\mu, \mu) &= \sup_{\|g\|_{\text{Lip}(\rho)} \leq 1} \sum_{x \in V} g(x) (f(x) - 1) \mu(x) \\ &= \frac{1}{2} \sup_{\|g\|_{\text{Lip}(\rho)} \leq 1} \sum_{x,y \in V} (g(x) - g(y)) (f(x) - f(y)) \mu(x) \mu(y) \\ &\leq \frac{1}{2} \sum_{x,y \in V} \rho(x, y) \sum_{\mathbf{e} \in \gamma_{xy}} |D_{\mathbf{e}}f| \mu(x) \mu(y) \\ &= \frac{1}{2} \sum_{\mathbf{e} \in E} |D_{\mathbf{e}}f| Q(\mathbf{e}) \cdot \frac{1}{Q(\mathbf{e})} \sum_{x,y \in V} 1_{\gamma_{xy}}(\mathbf{e}) \rho(x, y) \mu(x) \mu(y). \end{aligned}$$

Taking the expectation \mathbb{E}^γ , we obtain the desired result. \square

Corollary 3.6 (Weighted L^1 -Poincaré inequality). *Given a positive function φ on V , we have for any function f on V ,*

$$\int_V |f - \mu(f)| \varphi d\mu \leq \frac{\kappa}{2} \sum_{\mathbf{e} \in E} |D_{\mathbf{e}}f| Q(\mathbf{e}), \quad (3.11)$$

where κ is given by (3.10) with $\rho(x, y) = 1_{x \neq y}(\varphi(x) + \varphi(y))$, $x, y \in V$.

Proof. Considering $(f - c_1)/c_2$ if necessary, we assume without loss of generality that $f > 0$ and $\mu(f) = 1$. In that case, for the metric $\rho(x, y) = 1_{x \neq y}(\varphi(x) + \varphi(y))$, it is known that

$$W_{1,\rho}(f\mu, \mu) = \|\varphi(\nu - \mu)\|_{TV} = \int_V |f - 1| \varphi d\mu.$$

It remains to apply Theorem 3.5. \square

Remark 3.7. Taking $\varphi \equiv 1$ and considering the corresponding geometric constant κ , (3.11) is equivalent to (by Bobkov and Houdré [4, Theorem 1.1])

$$2\mu(A)\mu(A^c) \leq \kappa \sum_{\mathbf{e} \in \partial A} Q(\mathbf{e}), \quad \forall A \subset V,$$

where

$$\partial A := \{\mathbf{e} = (x, y) \in E; x \in A, y \in A^c\}$$

is the boundary of A . Thus (3.11) implies the standard Cheeger inequality

$$\mu(A) \leq \kappa \sum_{\mathbf{e} \in \partial A} Q(\mathbf{e}), \quad \forall A \subset V \text{ such that } \mu(A) \leq \frac{1}{2},$$

which has an equivalent functional version as : for every function f on V ,

$$\sum_x |f(x) - \text{med}_\mu(f)| \mu(x) \leq \frac{\kappa}{2} \sum_{\mathbf{e} \in E} |D_{\mathbf{e}} f| Q(\mathbf{e}), \quad (3.12)$$

where $\text{med}_\mu(f)$ is the median of f under μ . The Cheeger inequality (3.12) with the geometric constant κ is due to Diaconis and Stroock [13], whose idea goes back to Jerrum and Sinclair [19]. So Corollary 3.6 slightly improves theirs in this particular case.

The Cheeger isoperimetric inequality for general jump processes is studied by Chen and Wang [10].

Remark 3.8. If $G = (V, E)$ is a tree, i.e. there is only one path without circle from x to y for any two different vertices x, y , then the geometric constant κ becomes *optimal* for two types of metrics:

(a) $\rho(x, y) = 1_{x \neq y}(\varphi(x) + \varphi(y))$ (then the constant κ in the weighted L^1 -Poincaré inequality above is optimal in the case of trees);

(b) $\rho(x, y) = \rho_w(x, y)$, the distance induced by some length function w .

The optimality of κ for the two types of metrics in the case of trees is established by Liu-Ma-Wu [23], in a completely different way.

The usual Cheeger inequality exhibits the relationship between the isoperimetry and the Poincaré inequality ([1, 19, 21]). Now we present the relationship between the generalized Cheeger isoperimetric inequality and the Gaussian concentration.

Corollary 3.9. *Assume that $\sum_{y: y \sim x} q(x, y) \leq B$ for all $x \in V$. Then*

$$c_G \leq \kappa^2 B.$$

Proof. This is due to [18]. But for the self-completeness, we still present its proof. By the generalized Cheeger isoperimetric inequality in Theorem 3.5 and the Cauchy-Schwarz inequality, we have for any probability measure $\nu = f\mu$,

$$\begin{aligned} W_{1,\rho}(f\mu, \mu) &\leq \frac{\kappa}{2} \sum_{x \sim y} \mu(x) q(x, y) |f(x) - f(y)| \\ &\leq \kappa \sqrt{I(\nu|\mu)} \sqrt{\frac{1}{2} \sum_{x \sim y} \mu(x) q(x, y) (\sqrt{f(x)} + \sqrt{f(y)})^2} \\ &\leq \kappa \sqrt{I(\nu|\mu)} \sqrt{2 \sum_{x \sim y} \mu(x) q(x, y) f(x)} \\ &\leq \kappa \sqrt{I(\nu|\mu)} \sqrt{2B \sum_{x \in V} \mu(x) f(x)} = \kappa \sqrt{2BI(\nu|\mu)}, \end{aligned}$$

where the desired result follows. \square

Corollary 3.10. *For the Laplacian $\mathcal{L} = \Delta$ on the connected graph $G = (V, E)$, we have for the graph metric ρ_1 ,*

$$\kappa \leq \frac{d_*^2 b D}{|E|},$$

where d_*, D, b are given in Corollary 3.4.

Proof. Choosing γ_{xy} distributed uniformly on all shortest paths from x to y and $w = 1$, since $Q(\mathbf{e}) = 1/|E|, \mu(x) = d_x/|E|$, the geometric constant κ is bounded from above by

$$\max_{\mathbf{e} \in E} |E| \mathbb{E}^\gamma \sum_{x, y \in V} 1_{\gamma_{xy}}(\mathbf{e}) \rho_1(x, y) \left(\frac{d_*}{|E|} \right)^2 \leq \frac{d_*^2 b D}{|E|}.$$

□

4. SEVERAL EXAMPLES AND GRAPHS WITH SYMMETRY

4.1. Several examples. We begin with a baby-model.

Example 4.1 (Complete graph). Let $G = (V, E)$ be a complete graph with n different vertices, i.e. for any different $x, y \in V$, $(x, y) \in E$ ($n \geq 2$ of course). Consider the Laplacian $\mathcal{L} = \Delta$ and the graph metric ρ_1 which is now $\rho_1(x, y) = 1_{x \neq y}$. Hence μ is the uniform distribution on V and $Q(\mathbf{e}) = 1/|E| = 1/[n(n-1)]$. In such case the Dirichlet form is given by

$$\mathcal{E}(f, f) = \frac{1}{2} \sum_{\mathbf{e} \in E} (D_{\mathbf{e}} f)^2 Q(\mathbf{e}) = \frac{n}{n-1} \text{Var}_\mu(f),$$

where $\text{Var}_\mu(f) = \mu(f^2) - \mu(f)^2$ is the variance of f w.r.t. μ . So $c_P = \frac{n-1}{n}$.

In this example, we take $\gamma_{xy} = \{(x, y)\}$ as random paths and the length function $w \equiv 1$.

For the logarithmic Sobolev constant c_{LS} , notice that for any fixed edge $\mathbf{e} = (x_0, y_0)$, $\mathbf{e} \in \gamma_{xy}$ if and only if $x = x_0$ and $y = y_0$; and $L_{w, \mathbf{e}}(x) = 1_{x=x_0}/n$. Thus by Theorem 2.1,

$$\begin{aligned} c_{\text{LS}} &\leq |E| \left(\text{Ent}_\mu(L_{w, \mathbf{e}}) + \mu(L_{w, \mathbf{e}}) \log(e^2 + 1) \right) \\ &= n(n-1) \left[\frac{1}{n^2} \log \frac{1}{n} - \frac{1}{n^2} \log \frac{1}{n^2} + \frac{1}{n^2} \log(e^2 + 1) \right] \\ &= \left(1 - \frac{1}{n}\right) [\log n + \log(e^2 + 1)]. \end{aligned} \tag{4.1}$$

Comparing with the optimal logarithmic Sobolev constant $c_{\text{LS}} = \frac{n-1}{n-2} \log(n-1)$ for complete graph (see [12, Corollary A.5]), the estimate (4.1) has the correct order $\log n$.

Now we turn to bound the Gaussian concentration constant c_G by the geometric quantity K in (3.4), w.r.t. the graph metric ρ_1 . We have

$$K = \max_{(x, y) \in E} \frac{1}{Q(x, y)} \mu(x) \mu(y) = \frac{|E|}{n^2} = \frac{n-1}{n}.$$

Then by Theorem 3.1, for any μ -probability density f ,

$$W_{1,\rho_1}^2(f\mu, \mu) = \frac{1}{4} \left(\int_V |f - 1| d\mu \right)^2 \leq 2 \frac{n-1}{n} \mathcal{E}(\sqrt{f}, \sqrt{f}) = 2 \text{Var}_\mu(\sqrt{f}).$$

But by [17, Theorem 3.1], the corresponding optimal constant $c_G = \frac{n-1}{2n}$. Consequently, we have

$$\frac{K}{2} = \frac{n-1}{2n} = c_G \leq K.$$

For the generalized Cheeger isoperimetric inequality in Theorem 3.5 w.r.t. ρ_1 , we have

$$c_1 \leq \kappa = K \leq \frac{n-1}{n}$$

or equivalently

$$\sum_{x \in V} |f(x) - \mu(f)| \mu(x) \leq \frac{n-1}{n} \sum_{\mathbf{e} \in E} |D_{\mathbf{e}} f| Q(\mathbf{e}).$$

This inequality becomes equality for indicator function 1_A . Hence $c_1 = \kappa$, i.e. our generalized Cheeger isoperimetric inequality in Theorem 3.5 is optimal in this example.

Example 4.2 (Star). Consider a star $G = (V, E)$ with a central vertex v_0 and n outside vertices $\{v_i, i = 1, \dots, n\}$ connecting only with v_0 . For the Laplacian $\mathcal{L} = \Delta$, we have $\mu(v_0) = \frac{1}{2}, \mu(v_k) = \frac{1}{2n}, 1 \leq k \leq n$ and $Q(\mathbf{e}) = \frac{1}{2n}$ for every edge \mathbf{e} . It is known that $c_P = 1$ (c.f. [13]). Taking the length function $w \equiv 1$ in (2.4), we obtain by calculus

$$c_{LS} \leq \left(\frac{3}{2} - \frac{1}{n} \right) \log[2n(e^2 + 1)].$$

Applying the logarithmic Sobolev inequality to $f = 1_{v_k}, 1 \leq k \leq n$, we get $c_{LS} \geq \log(2n)/2$. Clearly, for large n , we have the correct order $\log n$.

For the geometric constants K and κ associated with the graph distance ρ_1 , taking γ_{xy} as the unique path from x to y without circle, we have

$$K \leq \frac{9}{2} - \frac{4}{n} \text{ and } \kappa = \frac{3}{2} - \frac{1}{n}.$$

Considering $f(v) = 2n1_{[v=v_1]}$,

$$W_1(f\mu, \mu) = \frac{3}{2} - \frac{1}{n}, \quad \sum_{\mathbf{e} \in E} |D_{\mathbf{e}} f| Q(\mathbf{e}) = 2,$$

we have that

$$c_1 = \kappa = \frac{3}{2} - \frac{1}{n},$$

i.e., the geometric quantity κ as an upper bound of c_1 is optimal.

Example 4.3 (Trees). Consider the full binary tree of depth d . For $d \geq 1$, such a tree has $2^{d+1} - 1$ vertices, $2^{d+1} - 2$ edges and the maximum degree is 3. Consider the Markov chain arising from nearest neighbor random walk on this tree. The longest

path is of length $2d$ and the value of b defined in (3.8) is $(2^d - 1)2^d$. By Corollary 3.4, we have

$$K \leq 18 \cdot 2^d d^3.$$

For κ w.r.t. the graph distance ρ_1 , by (3.10), Theorem 4.1 and Corollary 7.1 in [23],

$$\kappa = (2d - 3)2^d + 3 \leq 9 \cdot d2^{d-1},$$

which shows that Corollary 3.10 offers a good upper bound.

4.2. Graphs with symmetry. In this section, we shall consider various graphs with symmetry. See Chung [11] for examples and properties of symmetric graphs. For a graph $G = (V, E)$, an automorphism $f : V \rightarrow V$ is one-to-one mapping which preserves edges, i.e., for any $x, y \in V$, we have $(x, y) \in E$ if and only if $(f(x), f(y)) \in E$.

For any oriented edge $\mathbf{e} = (x, y) \in E$, consider the opposite oriented edge $\overleftarrow{\mathbf{e}} := (y, x)$ and the non-oriented edge $\mathbf{e}^0 := \{x, y\}$. Put $E^0 := \{\mathbf{e}^0; \mathbf{e} \in E\}$, the set of all non-oriented edges.

4.2.1. Edge-transitive graph. A graph G is edge-transitive if, for any two non-oriented edges $\{x, y\}, \{x', y'\}$, there is an automorphism f such that $\{f(x), f(y)\} = \{x', y'\}$.

Corollary 4.4. Assume that G is edge-transitive. For the Laplacian $\mathcal{L} = \Delta$ and the graph metric ρ_1 , we have

$$c_1 \leq \kappa \leq \mathbb{E} [\rho_1^2(X, Y)] \quad \text{and} \quad c_G \leq \kappa^2 \leq (\mathbb{E} [\rho_1^2(X, Y)])^2,$$

where the law of (X, Y) is $\mu \times \mu$ and μ is the uniform measure on V .

Proof. We consider a random (ordered) pair of vertices (X, Y) , chosen according to $\mu \times \mu$. Now given (X, Y) , we choose randomly a shortest path γ_{XY} between X and Y (uniformly chosen over all possible shortest paths from X to Y).

Notice that for $w \equiv 1$, $\rho = \rho_1$,

$$h(\mathbf{e}) := \frac{1}{Q(\mathbf{e})w(\mathbf{e})} \sum_{x, y \in V} \mathbb{E}^\gamma 1_{\gamma_{xy}}(\mathbf{e}) \rho(x, y) \mu(x) \mu(y) = |E| \mathbb{E} \rho_1(X, Y) 1_{\gamma_{XY}}(\mathbf{e})$$

satisfies $h(\mathbf{e}) = h(\overleftarrow{\mathbf{e}})$ for $\mathbf{e} \in E$ (that is true on any graph). Then h can be regarded as a function on E^0 .

Now by the edge-transitivity, $h(\mathbf{e})$ does not depend on \mathbf{e} , so we get by averaging over all edges \mathbf{e} ,

$$\begin{aligned} h(\mathbf{e}) &= \frac{1}{|E|} \sum_{\mathbf{e} \in E} |E| \mathbb{E} [\rho_1(X, Y) 1_{\gamma_{XY}}(\mathbf{e})] \\ &= \mathbb{E} [\rho_1^2(X, Y)]. \end{aligned}$$

Therefore

$$\kappa \leq \max_{\mathbf{e} \in E} h(\mathbf{e}) = \mathbb{E} [\rho_1^2(X, Y)].$$

Since $\sum_{y: y \sim x} q(x, y) \leq 1$ for all $x \in V$, we have $c_G \leq \kappa^2$ by Corollary 3.9. \square

Remark 4.5. From the result above, we may wonder whether on edge transitive graphs the correct order in diameter D of c_G is D^4 , and that of c_I is D^2 , which is indeed true. For example, for the Laplacian on the circle $\mathbb{Z}_p := \mathbb{Z}/p\mathbb{Z}$, c_P is of order D^2 . Taking the eigenfunction h corresponding to $\lambda_1 = 1/c_P$ with $\|h\|_{\text{Lip}(\rho_1)} = 1$, we see that $\text{Var}_\mu(h)$ is of order D^2 , too. By the central limit theorem,

$$\frac{1}{\sqrt{t}} \int_0^t h(X_s) ds$$

converges weakly to the normal law $N(0, \sigma^2(h))$, where the limit variance is

$$\sigma^2(h) = 2\langle (-\Delta)^{-1}h, h \rangle_\mu = 2c_P \text{Var}_\mu(h),$$

which is of order D^4 . But from the Gaussian concentration inequality we always have $c_G \geq \sigma^2(h)$. In other words c_G is at least of order D^4 . We leave to the reader for verifying that the correct order of c_I is D^2 .

Example 4.6 (Circle \mathbb{Z}_p). Let $p \in \mathbb{Z}$ and consider the integers mod p as p points around a circle. For x and y in \mathbb{Z}_p , choose γ_{xy} as the shorter of the two paths from x to y . For this model $\mu(x) = 1/p$ ($x \in \mathbb{Z}_p$), $Q(e) = 1/(2p)$ for any edge e . It is well known that $c_P = (1 - \cos \frac{2\pi}{p})^{-1}$, which is also the logarithmic Sobolev constant c_{LS} when p is even (c.f. [6]). Taking the length function $w \equiv 1$ in (2.4), we obtain by careful calculation,

$$\begin{aligned} c_{LS} &\leq \begin{cases} \frac{\log(3(e^2+1))}{12}(p+1)(p+2), & p \text{ is even;} \\ \frac{\log(3(e^2+1))}{12}(p+1)(p+2)(1+3/p), & p \text{ is odd;} \end{cases} \\ &\leq 5 \log(3(e^2+1))(1 - \cos \frac{2\pi}{p})^{-1}, \end{aligned} \quad (4.2)$$

which, together with $c_{LS} \geq c_P = (1 - \cos \frac{2\pi}{p})^{-1}$, offers a two sided estimate for c_{LS} with factor $5 \log(3(e^2+1))$.

Since the circle \mathbb{Z}_p is edge transitive, by Corollary 4.4, we have

$$c_I \leq \kappa \leq \frac{p^2}{12} + o(p^2); \quad c_G \leq \kappa^2 \leq \frac{p^4}{12^2} + o(p^4).$$

For this model, Sammer and Tetali [28] proved that the best constant in the transportation-entropy inequality (3.6) is $\frac{p^2}{48} + o(p^2)$. Then by Corollary 3.3, we have

$$\frac{p^4}{24^2} + o(p^4) \leq c_G \leq \frac{p^4}{12^2} + o(p^4).$$

Thus the correct order of c_G is p^4 , as mentioned in Remark 4.5.

4.2.2. Vertex-transitive graph. A graph G is vertex-transitive if, for any two vertices u and v there is an automorphism f such that $f(u) = v$. The automorphism group defines an equivalent relation on the edges of G . Two undirected edges $\mathbf{e}_1^0, \mathbf{e}_2^0$ are equivalent if and only if there is an automorphism π mapping \mathbf{e}_1^0 to \mathbf{e}_2^0 . We can

consider equivalent classes of undirected edges, denoted by E_1^0, \dots, E_s^0 . The index of G is defined as

$$\text{index}(G) = \max_i \frac{|E_i^0|}{|E_i^0|}.$$

Clearly $|E_i^0| \geq |V|, i = 1, \dots, s, 1 \leq \text{index}(G) \leq d$, where $d = d_x$ for any $x \in V$, the degree of the graph G . See [11].

For any edge \mathbf{e} such that $\mathbf{e}^0 \in E_i^0, i = 1, \dots, s$

$$\begin{aligned} \mathbb{E}^{\mu \times \mu} [\rho_1(X, Y) 1_{\{\mathbf{e} \in \gamma_{XY}\}}] &= \frac{1}{2|E_i^0|} \sum_{\mathbf{e}: \mathbf{e}^0 \in E_i^0} \mathbb{E}^{\mu \times \mu} [\rho_1(X, Y) 1_{\{\mathbf{e} \in \gamma_{XY}\}}] \\ &\leq \frac{\text{index}(G)}{|E|} \mathbb{E}^{\mu \times \mu} [\rho_1^2(X, Y)]. \end{aligned}$$

For any two vertices $y, y' \in V$, there is an automorphism such that $f(y) = y'$. Since the stationary distribution $\mu \equiv 1/|E|$ and $\rho_1(x, f(y)) = \rho_1(f^{-1}(x), y)$ for any $x \in V$, we have

$$\mathbb{E}^{\mu} [\rho_1^2(X, y')] = \mathbb{E}^{\mu} [\rho_1^2(X, f(y))] = \mathbb{E}^{\mu} [\rho_1^2(f^{-1}(X), y)] = \mathbb{E}^{\mu} [\rho_1^2(X, y)].$$

Then for any fixed vertex $v_0 \in V$,

$$\mathbb{E}^{\mu \times \mu} [\rho_1^2(X, Y)] = \mathbb{E}^{\mu} [\rho_1^2(X, v_0)].$$

Therefore, we have proved the following estimate of the Gaussian concentration constant c_G in (3.1).

Corollary 4.7. *Let Δ be the Laplacian operator on a vertex transitive graph G . For any fixed vertex $v_0 \in V$ and the graph metric ρ_1 , we have*

$$c_1 \leq \kappa \leq \text{index}(G) \mathbb{E}^{\mu} [\rho_1^2(X, v_0)] \leq d \mathbb{E}^{\mu} [\rho_1^2(X, v_0)]$$

and

$$c_G \leq K \wedge \kappa^2,$$

where

$$K \leq \text{index}(G) \mathbb{E}^{\mu} [\rho_1^4(X, v_0)] \leq d \mathbb{E}^{\mu} [\rho_1^4(X, v_0)].$$

The proof of the bound on K is similar, omitted here.

4.2.3. Distance transitive graph. A graph G is distance transitive if, for any two pairs of vertices $\{x, y\}, \{x', y'\}$ with $\rho_1(x, y) = \rho_1(x', y')$, there is an automorphism mapping x to x' and y to y' . The distance transitive graph is both edge-transitive and vertex-transitive. Then $\text{index}(G) = 1$. The estimate in Corollary 4.7 holds for distance transitive graphs, that is, for any fixed vertex $v_0 \in V$,

$$c_G \leq (\mathbb{E}^{\mu} [\rho_1^2(X, v_0)])^2 \quad \text{and} \quad c_1 \leq \mathbb{E}^{\mu} [\rho_1^2(X, v_0)]. \quad (4.3)$$

Example 4.8. The vertices set V consists of all the subsets of k elements in $\{1, 2, \dots, n\}$ ($1 \leq k \leq n$ is fixed). Define a metric on V by $d(x, y) = k - |x \cap y|$. The edges set is given by $\{(x, y) \in V^2; d(x, y) = 1\}$. This is a distance transitive graph. The Markov process generated by the Laplacian Δ on G is known as the Bernoulli-Laplace diffusion model. See Lee and Yau [22] for the estimate of the logarithmic Sobolev constant, Gao and Quastel [16] for the exponential decay rate of entropy.

By (4.3), we have for the graph metric $\rho_1 = d$,

$$c_1 \leq \kappa \text{ and } c_G \leq \kappa^2,$$

where

$$\kappa \leq \frac{1}{\binom{n}{k}} \sum_{j=0}^{\min\{k, n-k\}} j^2 \binom{k}{j} \binom{n-k}{j}.$$

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